

An Araki-Lieb-Thirring inequality for geometrically concave and geometrically convex functions

Koenraad M.R. Audenaert

*Department of Mathematics, Royal Holloway, University of London,
Egham TW20 0EX, United Kingdom*

Abstract

For positive definite matrices A and B , the Araki-Lieb-Thirring inequality amounts to an eigenvalue log-submajorisation relation for fractional powers

$$\lambda(A^t B^t) \prec_{w(\log)} \lambda^t(AB), \quad 0 < t \leq 1,$$

while for $t \geq 1$, the reversed inequality holds. In this paper I generalise this inequality, replacing the fractional powers x^t by a larger class of functions. Namely, a continuous, non-negative, geometrically concave function f with domain $\text{dom}(f) = [0, x_0]$ for some positive x_0 (possibly infinity) satisfies

$$\lambda(f(A)f(B)) \prec_{w(\log)} f^2(\lambda^{1/2}(AB)),$$

for all positive semidefinite A and B with spectrum in $\text{dom}(f)$, if and only if $0 \leq xf'(x) \leq f(x)$ for all $x \in \text{dom}(f)$. The reversed inequality holds for continuous, non-negative, geometrically convex functions if and only if they satisfy $xf'(x) \geq f(x)$ for all $x \in \text{dom}(f)$. As an application I derive a complementary inequality to the Golden-Thompson inequality.

Key words: Log-majorisation, positive semidefinite matrix, matrix inequality
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1 Introduction

The Araki-Lieb-Thirring (ALT) inequality [2,7] states that for $0 < t \leq 1$ and positive definite matrices A and B , the eigenvalues of $A^t B^t$ are log-submajorised by the eigenvalues of $(AB)^t$. For $n \times n$ matrices X and Y with

Email address: `koenraad.audenaert@rhul.ac.uk` (Koenraad M.R. Audenaert).

positive spectrum, the log-submajorisation relation $\lambda(X) \prec_{w(\log)} \lambda(Y)$ means that for all $k = 1, \dots, n$, the following holds:

$$\prod_{j=1}^k \lambda_j(X) \leq \prod_{j=1}^k \lambda_j(Y).$$

This is equivalent to weak majorisation of the logarithms of the spectra

$$\log \lambda(X) \prec_w \log \lambda(Y).$$

Here and elsewhere, I adhere to the convention to sort eigenvalues in non-increasing order; that is, $\lambda_1(X) \geq \lambda_2(X) \geq \dots \geq \lambda_n(X)$.

With this notation, the ALT inequality can be written as

$$\lambda(A^t B^t) \prec_{w(\log)} \lambda^t(AB), \quad 0 < t \leq 1. \quad (1)$$

For positive scalars a and b , (1) reduces to the equality $a^t b^t = (ab)^t$.

One can ask whether similar inequalities hold for other functions than the fractional powers x^t . One possibility is to consider functions that satisfy

$$\lambda(f(A)f(B)) \prec_{w(\log)} f(\lambda(AB)).$$

As the scalar case reduces to $f(a)f(b) \leq f(ab)$ these functions must be super-multiplicative. Another possibility, and the one pursued here, is to consider functions satisfying

$$\lambda(f(A)f(B)) \prec_{w(\log)} f^2\left(\sqrt{\lambda(AB)}\right). \quad (2)$$

Here, the scalar case reduces to $f(a)f(b) \leq f^2(\sqrt{ab})$, for all $a, b > 0$. Functions satisfying this requirement are called *geometrically concave* (see Definition 1 below). In this paper I completely characterise the class of geometrically concave functions that satisfy (2) for all positive definite matrices A and B .

Likewise, as inequality (1) holds in the reversed sense for $t \geq 1$, one may ask for which functions f the reversed inequality holds for all positive definite matrices A and B :

$$f^2\left(\sqrt{\lambda(AB)}\right) \prec_{w(\log)} \lambda(f(A)f(B)). \quad (3)$$

Here the scalar case restricts the class of functions to those satisfying the relation $f^2(\sqrt{ab}) \leq f(a)f(b)$. Such functions are called *geometrically convex*.

I also completely characterise the class of geometrically convex functions that satisfy (3) for all positive definite matrices A and B .

The concepts of geometric concavity and geometric convexity were first studied by Montel [8] and have recently received attention from the matrix community [3,5].

Definition 1 *Let I be the interval $I = [0, x_0)$, with $x_0 > 0$ (possibly infinite). A function $f : I \rightarrow [0, \infty)$ is geometrically concave if for all $x, y \in I$, $\sqrt{f(x)f(y)} \leq f(\sqrt{xy})$. It is geometrically convex if for all $x, y \in I$, $\sqrt{f(x)f(y)} \geq f(\sqrt{xy})$.*

Equivalently, a function $f(x)$ is geometrically concave (convex) if and only if the associated function $F(y) := \log(f(e^y))$ is concave (convex).

The main results of this paper are summarised in the next section, the proofs of the main theorems (Theorems 1 and 2) are given in Section 3, and the paper concludes with a brief application in Section 4.

2 Main Results

To state the main results of this paper most succinctly, let us define two classes of functions.

Definition 2 *A continuous non-negative function f with domain an interval $I = [0, x_0)$ is in class \mathcal{A} if and only if it is geometrically concave and its derivative f' satisfies $0 \leq xf'(x) \leq f(x)$ for all $x \in I$ where the derivative exists.*

Definition 3 *A continuous non-negative function f with domain an interval $I = [0, x_0)$ is in class \mathcal{B} if and only if it is geometrically convex and its derivative f' satisfies $xf'(x) \geq f(x)$ for all $x \in I$ where the derivative exists.*

In terms of the associated function $F(y) = \log(f(\exp y))$, $f \in \mathcal{A}$ if and only if $F(y)$ is concave and $0 \leq F'(y) \leq 1$ for all y where F is differentiable, and $f \in \mathcal{B}$ if and only if $F(y)$ is convex and $1 \leq F'(y)$ for all y where F is differentiable.

There is a simple one-to-one relationship between these two classes; essentially f is in class \mathcal{A} if and only if its inverse function f^{-1} is in class \mathcal{B} . However, some care must be taken as \mathcal{A} contains the constant functions and also those functions that are constant on some interval.

Proposition 1 *A function f that is non-constant on the interval $[0, x_1] \subseteq [0, x_0)$ is in class \mathcal{A} if and only if the inverse of the restriction of f to $[0, x_1]$ is in class \mathcal{B} .*

Proof. It is clear that a concave monotonous function f is always invertible over the entire interval where it is not constant. We will henceforth identify the inverse of f with the inverse of the restriction of f on that interval.

If $F(y)$ is the associated function of $f(x)$ then the associated function of f^{-1} is the inverse function of F , F^{-1} . Now f is in class \mathcal{A} if and only if F is concave, monotonous and $F' \leq 1$. This implies that the inverse function $G = F^{-1}$ is convex and satisfies $G' \geq 1$, which in turn implies that G is the associated function of a function g in class \mathcal{B} . This shows that $f \in \mathcal{A}$ implies $f^{-1} \in \mathcal{B}$.

A similar argument reveals that the converse statement holds as well. \square

The main result of this paper is the following theorem:

Theorem 1 *Let f be a continuous non-negative function with domain an interval $I = [0, x_0)$, $x_0 > 0$ (possibly infinite), then*

$$\lambda(f(A)f(B)) \prec_{w(\log)} f^2\left(\sqrt{\lambda(AB)}\right) \quad (4)$$

holds for all positive definite matrices A and B with spectrum in I if and only if f is in class \mathcal{A} .

That the right-hand side of (4) is well-defined follows from the following lemma:

Lemma 1 *If A and B are positive semidefinite matrices with eigenvalues in the interval $I = [a, b]$, $0 \leq a < b$, the positive square roots of the eigenvalues of AB are in I as well.*

Proof. We have $a \leq A, B \leq b$, which implies

$$a^2 \leq aA \leq A^{1/2}BA^{1/2} \leq bA \leq b^2.$$

Hence, $a^2 \leq \lambda_i(A^{1/2}BA^{1/2}) \leq b^2$, so that $a \leq \lambda_i^{1/2}(AB) \leq b$. \square

A simple consequence of Theorem 1 is that the reversed inequality holds if and only if f is in class \mathcal{B} .

Theorem 2 *Let g be a continuous non-negative function with domain an interval $I = [0, x_0)$, $x_0 > 0$ (possibly infinite), then*

$$g^2\left(\sqrt{\lambda(XY)}\right) \prec_{w(\log)} \lambda(g(X)g(Y)). \quad (5)$$

holds for all positive definite matrices X and Y with spectrum in I if and only if g is in class \mathcal{B} .

3 Proofs

We now turn to the proofs of Theorems 1 and 2.

3.1 Proof of necessity

To show necessity of the conditions $f \in \mathcal{A}$ ($f \in \mathcal{B}$) I consider two special 2×2 matrices with eigenvalues a and b , $0 \leq b < a$, such that $a \in \text{dom}(f)$ and f is differentiable in a :

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad B = U \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} U^*, \quad \text{with } U = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

We will consider values of θ close to 0. The largest eigenvalue of AB can be calculated in a straight-forward fashion. The quantity $f^2(\sqrt{\lambda_1(AB)})$ can then be expanded in a power series of the variable θ . To second order this yields

$$f^2(\sqrt{\lambda_1(AB)}) = f(a)^2 - \frac{a(a-b)f(a)f'(a)}{a+b}\theta^2 + O(\theta^4).$$

In a similar way we also get

$$\lambda_1(f(A)f(B)) = f(a)^2 - \frac{f(a)^2(f(a) - f(b))}{f(a) + f(b)}\theta^2 + O(\theta^4).$$

Hence, to satisfy inequality (4), the following must be satisfied for all $0 \leq b < a \in \text{dom}(f)$:

$$\frac{a(a-b)f(a)f'(a)}{a+b} \leq \frac{f(a)^2(f(a) - f(b))}{f(a) + f(b)}.$$

In particular, take $b = 0$. As f has to be geometrically concave, $f(0) = 0$. The condition then becomes

$$af(a)f'(a) \leq f(a)^2, \forall a \in \text{dom}(f),$$

which reduces to the defining condition for $f \in \mathcal{A}$.

Necessity of the condition $xf'(x) \geq f(x)$ in Corollary 2 also follows immediately from this special pair of matrices.

Note that in the preceding proof we see why the domain of f should include the point $x = 0$.

3.2 Proof of sufficiency for Theorem 1

Now I turn to proving sufficiency of the condition $f \in \mathcal{A}$. The main step consists in showing that the set of functions f for which the inequality (4) holds is ‘geometrically convex’; that is, the set of associated functions F for these f is convex. To show this a number of preliminary propositions are needed.

Lemma 2 *Let R_1, R_2, S_1 and S_2 be positive semidefinite matrices such that R_1 commutes with R_2 and S_1 with S_2 . Let $R = R_1^{1/2} R_2^{1/2}$ and $S = S_1^{1/2} S_2^{1/2}$. Then*

$$\lambda_1((R^{1/2} S R^{1/2})^2) \leq \lambda_1(R_1^{1/2} S_1 R_1^{1/2} R_2^{1/2} S_2 R_2^{1/2}). \quad (6)$$

Proof. We will prove this by showing that the equality

$$\lambda_1(R_1^{1/2} S_1 R_1^{1/2} R_2^{1/2} S_2 R_2^{1/2}) = a$$

implies the inequality $\lambda_1((R^{1/2} S R^{1/2})^2) \leq a$.

W.l.o.g. we can assume that the matrices R_1 and S_1 are invertible; then the equality indeed leads to the following sequence of implications:

$$\begin{aligned} & \lambda_1(R_1^{1/2} S_1 R_1^{1/2} R_2^{1/2} S_2 R_2^{1/2}) = a \\ \implies & (R_1^{1/2} S_1 R_1^{1/2})^{1/2} R_2^{1/2} S_2 R_2^{1/2} (R_1^{1/2} S_1 R_1^{1/2})^{1/2} \leq a \\ \implies & R_2^{1/2} S_2 R_2^{1/2} \leq a R_1^{-1/2} S_1^{-1} R_1^{-1/2} \\ \implies & S_1^{1/2} R_1^{1/2} R_2^{1/2} S_2 R_2^{1/2} R_1^{1/2} S_1^{1/2} = S_1^{1/2} R S_2 R S_1^{1/2} \leq a \\ \implies & \sigma_1(S_2^{1/2} R S_1^{1/2}) \leq \sqrt{a} \\ \implies & |\lambda_1(S_2^{1/2} R S_1^{1/2})| \leq \sqrt{a}. \end{aligned}$$

The last implication is the simplest case of Weyl’s majorant theorem.

Now note that $S_2^{1/2} R S_1^{1/2}$ and $S_1^{1/2} R S_2^{1/2} = S_1^{1/4} S_2^{1/4} R S_1^{1/4} S_2^{1/4}$ have the same non-zero eigenvalues. Hence, $\lambda_1(S_1^{1/2} R S_2^{1/2}) = \lambda_1(R^{1/2} S R^{1/2}) \leq \sqrt{a}$, and the inequality $\lambda_1((R^{1/2} S R^{1/2})^2) \leq a$ follows. \square

Proposition 2 *Under the conditions of Lemma 2,*

$$\prod_{i=1}^k \lambda_i(R_1^{1/2} R_2^{1/2} S_1^{1/2} S_2^{1/2}) \leq \prod_{i=1}^k \lambda_i^{1/2}(R_1 S_1) \lambda_i^{1/2}(S_2 R_2). \quad (7)$$

Proof. Since each side of (6) is the largest eigenvalue of a product of powers of matrices, we can use the well-known Weyl trick of replacing every matrix by its antisymmetric tensor power to boost the inequality to the log-submajorisation relation

$$\prod_{i=1}^k \lambda_i((R_1^{1/2} S R_1^{1/2})^2) \leq \prod_{i=1}^k \lambda_i(R_1^{1/2} S_1 R_1^{1/2} R_2^{1/2} S_2 R_2^{1/2}).$$

Combining this with Lidskii's inequality $\prod_{i=1}^k \lambda_i(AB) \leq \prod_{i=1}^k \lambda_i(A) \lambda_i(B)$ ([4], Corollary III.4.6), valid for positive definite A and B , and then taking square roots yields inequality (7). \square

Inequality (7) can be interpreted as midpoint geometric convexity of the function

$$p \mapsto f_k(p) = \prod_{i=1}^k \lambda_i(R_1^p R_2^{1-p} S_1^p S_2^{1-p});$$

that is, $f_k(1/2) \leq \sqrt{f_k(1)f_k(0)}$. We now use a standard argument (see e.g. the proof of Lemma IX.6.2 in [4]) to show that this actually implies geometric convexity in full generality, i.e. $f_k(p) \leq f_k(1)^p f_k(0)^{1-p}$ for all $p \in [0, 1]$.

Proposition 3 *Under the conditions of Lemma 2, and for all $p \in [0, 1]$,*

$$\prod_{i=1}^k \lambda_i(R_1^p R_2^{1-p} S_1^p S_2^{1-p}) \leq \prod_{i=1}^k \lambda_i^p(R_1 S_1) \lambda_i^{1-p}(S_2 R_2). \quad (8)$$

Proof. By Proposition 2 the inequality holds for $p = 1/2$. It trivially holds for $p = 0$ and $p = 1$.

Let $s, t \in [0, 1]$ be given. Applying Proposition 2 with the matrices R_1, S_1, R_2 and S_2 replaced by $R_1^t R_2^{1-t}, S_1^t S_2^{1-t}, R_1^s R_2^{1-s}$ and $S_1^s S_2^{1-s}$, respectively, yields the inequality

$$\begin{aligned} & \prod_{i=1}^k \lambda_i(R_1^{(s+t)/2} R_2^{1-(s+t)/2} S_1^{(s+t)/2} S_2^{1-(s+t)/2}) \\ & \leq \prod_{i=1}^k \lambda_i^{1/2}(R_1^t R_2^{1-t} S_1^t S_2^{1-t}) \lambda_i^{1/2}(R_1^s R_2^{1-s} S_1^s S_2^{1-s}). \end{aligned}$$

Now assume that the inequality (8) holds for the values $p = s$ and $p = t$. Thus

$$\begin{aligned}
& \prod_{i=1}^k \lambda_i^{1/2}(R_1^t R_2^{1-t} S_1^t S_2^{1-t}) \lambda_i^{1/2}(R_1^s R_2^{1-s} S_1^s S_2^{1-s}) \\
& \leq \prod_{i=1}^k \lambda_i^{t/2}(R_1 S_1) \lambda_i^{(1-t)/2}(S_2 R_2) \lambda_i^{s/2}(R_1 S_1) \lambda_i^{(1-s)/2}(S_2 R_2) \\
& = \prod_{i=1}^k \lambda_i^{(s+t)/2}(R_1 S_1) \lambda_i^{1-(s+t)/2}(S_2 R_2).
\end{aligned}$$

In other words, the assumption that (8) holds for the values $p = s$ and $p = t$ implies that it also holds for their midpoint $p = (s + t)/2$.

Using induction this shows that (8) holds for all dyadic rational values of p (i.e. rationals of the form $k/2^n$, with k and n integers such that $k \leq 2^n$). Invoking continuity and the fact that the dyadic rationals are dense in $[0, 1]$, this finally implies that (8) holds for all real values of p in $[0, 1]$. \square

We are now ready to prove our first intermediate result: convexity of the set of associated functions F for which the inequality (4) holds.

Proposition 4 *Let $f_1(x)$ and $f_2(x)$ be two continuous, non-negative functions with domain an interval I of the non-negative reals, and for which (4) holds for all positive semidefinite A and B with spectrum in I . Let $p \in [0, 1]$ and let $f(x) = f_1^p(x) f_2^{1-p}(x)$. Then (4) holds for f too.*

Proof. Let us fix the matrices A and B and let $R_i = f_i(A)$ and $S_i = f_i(B)$, $i = 1, 2$. These matrices R_i and S_i clearly satisfy the conditions of Proposition 3 (positivity and commutativity). Hence

$$\begin{aligned}
\prod_{i=1}^k \lambda_i(f(A) f(B)) &= \prod_{i=1}^k \lambda_i(f_1^p(A) f_2^{1-p}(A) f_1^p(B) f_2^{1-p}(B)) \\
&\leq \prod_{i=1}^k \lambda_i^p(f_1(A) f_1(B)) \lambda_i^{1-p}(f_2(A) f_2(B)).
\end{aligned}$$

By the assumption that f_1 and f_2 satisfy inequality (4), this implies

$$\begin{aligned}
\prod_{i=1}^k \lambda_i(f(A) f(B)) &\leq \prod_{i=1}^k f_1^{2p}(\lambda_i^{1/2}(AB)) f_2^{2(1-p)}(\lambda_i^{1/2}(AB)) \\
&= \prod_{i=1}^k f^2(\lambda_i^{1/2}(AB)),
\end{aligned}$$

i.e. f satisfies inequality (4) as well. \square

We have already proven that membership of this class is a necessary condition for inequality (4) to hold. The set of associated functions F for functions in class \mathcal{A} is the set of concave functions F that satisfy $0 \leq F'(y) \leq 1$ for all y in the domain of F where F is differentiable. This set is convex, as can be seen from the fact that, for $f \in \mathcal{A}$, F' is non-increasing and the range of F' is $[0, 1]$. Hence, F' is a convex combination of step functions $\Phi(b - y)$ (with Φ the Heaviside step function) and the constant functions 0 and 1: $F'(y) = r + s \int_{(-\infty, +\infty)} \Phi(b - y) d\mu(b)$, where $r, s \geq 0$, $r + s \leq 1$, and $d\mu$ is a probability measure (normalised positive measure). Hence, such F have the integral representation

$$F(y) = \alpha + ry + s \int_{(-\infty, +\infty)} \min(y, t) d\mu(t). \quad (9)$$

The additive constant α corresponds to multiplication of f by e^α , so we may assume that $\alpha = 0$. Since $r + s \leq 1$ it then follows that f is in the geometric convex closure of $f(x) = 1$, $f(x) = x$ and $f(x) = \min(x, c)$ for $c \in I$ ($c = e^t$).

The next step of the proof is to show that inequality (4) holds for these extremal functions. For the functions $f(x) = 1$ and $f(x) = x$ this is of course trivial to prove. Hence let us consider the remaining function $f(x) = \min(x, c)$, with $c \in I$. As the constant c can be absorbed in the matrices A and B , we only need to check the function $f(x) = \min(x, 1)$. The action of this function on a matrix A is to replace any eigenvalue of A that is bigger than 1 by the value 1. I denote this matrix function by $\min(A, 1)$. For this function a stronger inequality can be proven than what is actually needed.

Lemma 3 *For $A, B \geq 0$, and for any i*

$$\lambda_i(\min(A, 1) \min(B, 1)) \leq \min(\lambda_i(AB), 1).$$

Proof. Let $A_1 = \min(A, 1)$ and $B_1 = \min(B, 1)$. We have $A_1 \leq A$ and $B_1 \leq B$, so that, using Weyl monotonicity of the eigenvalues twice,

$$\begin{aligned} \lambda_i(A_1 B_1) &= \lambda_i(A_1^{1/2} B_1 A_1^{1/2}) \\ &\leq \lambda_i(A_1^{1/2} B A_1^{1/2}) \\ &= \lambda_i(B^{1/2} A_1 B^{1/2}) \\ &\leq \lambda_i(B^{1/2} A B^{1/2}) \\ &= \lambda_i(AB). \end{aligned}$$

This implies also that $\min(\lambda_i(A_1 B_1), 1) \leq \min(\lambda_i(AB), 1)$.

We also have $A_1 \leq \mathbf{I}$ and $B_1 \leq \mathbf{I}$, hence by Lemma 1 $\min(\lambda_i(A_1 B_1), 1) = \lambda_i(A_1 B_1)$. \square

Since the inequality of this lemma implies the weaker log-submajorisation inequality

$$\lambda(\min(A, 1) \min(B, 1)) \prec_{w(\log)} \min(\lambda(AB), 1),$$

all extremal points of the class \mathcal{A} satisfy the inequality (4).

Finally, by Proposition 4 this implies that (4) holds for all functions in \mathcal{A} , hence membership of \mathcal{A} is a sufficient condition. This ends the proof of Theorem 1. \square

3.3 Proof of sufficiency for Theorem 2.

Let $A = g(X)$ and $B = g(Y)$, with $g = f^{-1}$. Thus, $X = f(A)$ and $Y = f(B)$. Since f is in \mathcal{A} , g is in \mathcal{B} . Inequality (4) then gives

$$\lambda(XY) \prec_{w(\log)} f^2 \left(\sqrt{\lambda(g(X)g(Y))} \right). \quad (10)$$

The right-hand side features the function $w(x) = f^2(\sqrt{x})$. Because f is geometrically concave, so is w . The inverse function w^{-1} is given by $w^{-1}(y) = g^2(\sqrt{y})$. Therefore, w^{-1} is geometrically convex. Furthermore, because f' is non-negative, w^{-1} is monotonously increasing.

A monotonous convex function preserves the weak majorisation relation ([4], Corollary II.3.4). Thus, a monotonous geometrically convex function preserves the log-submajorisation relation. Hence, when w^{-1} is applied to both sides of (10) one obtains

$$w^{-1}(\lambda(XY)) \prec_{w(\log)} \lambda(g(X)g(Y)),$$

which is (5). \square

4 Application

An interesting application concerns the function $f(x) = 1 - \exp(-x)$, which is in class \mathcal{A} . A simple application of Theorem 1 leads to an inequality that is complementary to the famous Golden-Thompson inequality $\text{Tr} \exp(A + B) \leq \text{Tr} \exp(A) \exp(B)$ (where A and B are Hermitian).

In [6] (see also [1]) the inequality

$$\text{Tr}(\exp(pA) \# \exp(pB))^{2/p} \leq \text{Tr} \exp(A + B)$$

was proven, for every $p > 0$ and Hermitian A and B . This is complementary to the Golden-Thompson inequality because it provides a lower bound on $\text{Tr} \exp(A + B)$. The bound obtained below is complementary in a different sense, as it provides an upper bound on $\text{Tr} e^{-A} e^{-B}$ (for *positive* A and B).

Theorem 3 For $A, B \geq 0$, and with $C = (A^{1/2} B A^{1/2})^{1/2}$,

$$\text{Tr}(e^{-A} e^{-B}) \leq \text{Tr}(e^{-A} + e^{-B}) + \text{Tr}(e^{-2C} - 2e^{-C}). \quad (11)$$

Proof. The inequality can be rewritten as $\text{Tr} f(A) f(B) \leq \text{Tr} f^2(C)$, with $f(x) = 1 - e^{-x}$. By Lemma 5 in [3] $f(x)$ is geometrically concave. Moreover, f is in \mathcal{A} : obviously, $f' \geq 0$; secondly, $f'(x) = \exp(-x) \leq 1/(1+x)$, so that $x f'(x) = x \exp(-x) \leq 1 - \exp(-x) = f(x)$.

Hence f satisfies the conditions of Theorem 1. The inequality follows immediately from that theorem, as log-submajorisation implies weak majorisation, and majorisation of the trace, in particular. \square

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